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Representations of the Virasoro algebra by the orbit method

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *A geometric background is given for representation with a highest weight of the Virasoro algebra. The representation space consists of holomorphic sections of an analytic line bundle over the manifold* $M = \mathrm{Diff}_+ S^1 / \mathrm{Rot}\, S^1$ *or over its factor manifold* $M_1 = \mathrm{Diff}_+ S^1/\mathrm{PSL}(2, \mathbf{R})$.

A class of polynomial sections of such a bundle is introduced and explicit formulae of the action are written. We investigate also an equivariant embedding of M_1 *in the infinite dimensional analog of the complex symmetric domain of type III.*

INTRODUCTION

The aim of this paper is to give a geometric background for representations with a highest weight of the Virasoro algebra and the Virasoro-Bott group.

The representation space consists of holomorphic sections of an analytic line bundle with a connection over the manifold $M = \mathrm{Diff}_+ S^1 / \mathrm{Rot}\, S^1$ or over its factor manifold $M_1 = \mathrm{Diff}_+ S^1/\mathrm{PSL}(2, \mathbf{R})$.

A class of polynomial sections of such a bundle is introduced in which the Virasoro algebra acts and which contains the coefficients of an equivariant imbedding of M or M_1 in the infinite dimensional analog of a complex symmetric homogeneous domain of type III.

Key-Words: Representation of the Virasoro algebra with a highest weight infinited dimensional Kähler manifold geometric quantization.

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I. THE GEOMETRY OF $M = \mathrm{Diff}_+ S^1 / \mathrm{Rot}\, S^1$

Let $\mathrm{Diff}_+ S^1$ denote the group of diffeomorphisms of the unit circle S^1 preserving an orientation. We shall parametrize S^1 by complex numbers z with $|z| = 1$. An element $\zeta \in \mathrm{Diff}_+ S^1$ has the form $\zeta(e^{i\theta}) = e^{i\varphi(\theta)}$ where φ is a smooth real function satisfying the condition $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi, \varphi'(\theta) > 0$. The Lie algebra of this group is identified with the algebra $\mathrm{Vect}\, S^1$ or smooth vector fields on the circle. The complex span of $\mathrm{Vect}\, S^1$ will be denoted by $\mathbb{C}\ \mathrm{Vect}\, S^1$. It has natural basis $e^k = ie^{ikt}\mathrm{d}/\mathrm{d}t,\ k \in \mathbb{Z}$.

It is well know since [1] that this Lie algebra has essentially unique non-trivial central extension defined by the 2-cocycle

$$c(V_1, V_2) = \frac{1}{2\pi}\int_0^{2\pi} V_1'(t)\,\mathrm{d}\,V_2'(t)$$

This extension appeared independently in [2] and is known as the Virasoro algebra. More presisely we denote by Cvir the complex linear span of e^k and the central element Z with commutation relations

$$[e_k, e_l] = (k - l)e_{k+l} + \delta_{k,-l}\,\frac{k^3 - k}{12}Z$$

We introduce the antilinear involution in Cvir putting $e_k^+ = e_{-k}, Z^+ = Z$, and denote by vir the set of antihermitian elements in Cvir. Then vir will be a central extension of the Lie algebra $\mathrm{Vect}^{pol}\, S^1 = \mathbf{R}[\cos t, \sin t]\frac{\mathrm{d}}{\mathrm{d}t}$ consisting of real polynomial vector fields on S^1. The corresponding infinite dimensional Lie group Vir is a central extension of the group $\mathrm{Diff}_+ S^1$ and is defined by the cocycle found by Bott [3]:

$$c(\zeta_1, \zeta_2) = \frac{1}{2\pi}\int_0^{2\pi} \log(\zeta_1' \circ \zeta_2)\,\mathrm{d}\log \zeta_2'$$

We shall call it the Virasoro-Bott group.

By the flag manifold for the Virasoro-Bott group we mean the homogeneous space $M = \mathrm{Diff}_+ S^1/\mathrm{Rot}\, S^1$ where $\mathrm{Rot}\, S^1$ is the one-dimensional group of rigid rotations of S^1.

This manifold arises in many ways as an orbit in the coadjoint representation of groups $\mathrm{Diff}_+ S^1$ and Vir ([4], [5]). It follows that M admits a two-parameter family of sympletic structures $\omega_{h,c};\ h, c \in \mathbf{R}$.

Now let S be the set of univalent holomorphic functions in the unit disc ([6]). It is an infinite dimensional complex manifold with natural coordinates defined by $f(z) = z(1 + \sum_{k=1}^{\infty} c_k z^k)$ In [7], [8] it is shown that one can define a transitive action of the group $\mathrm{Diff}_+ S^1$ via holomorphisms of such that the stabilizer of the point $f_0(z) = z$ be the subgroup Rot S^1. Hence, S can be identified with M and defines the complex structure on M. Together with the sympletic structure $\omega_{h,c}$ this complex structure generates the pseudokählerian metric $\omega_{h,c}$. In the initial point f_0 this metric has the form

$$w_{h,c} = \sum_{k=1}^{\infty}[2hk + \frac{c}{12}(k^3 - k)]\, d c_k d \bar{c}_k$$

Note that we use here a different parametrisation compared with [9], [10] (Where we used parameters $\alpha = 2h - \frac{c}{12}$, $\beta = \frac{c}{12}$).

2. ANALYTIC LINE BUNDLES $E_{h,c}$ OVER M

Here we construct for each Diff$_+$ S^1-invariant pseudokählerian metric $w_{h,c} = \sum_{k,l} w_{h,c}^{k,l}(c_1, c_2 \ldots) d c_k d \bar{c}_l$, one-dimensional complex vector-bundle $E_{h,c}$ over M with the following properties:

1. $E_{h,c}$ is an analytic line bundle over complex manifold M.
2. The Lie algebra CVet S^1 or its central extension Cvir acts on $E_{h,c}$ by holomorphic fibrewise linear vector fields.
3. There are invariant hermitian metric g and hermitian connection ∇ on $E_{h,c}$.
4. The curvature form of the connection ∇ is equal to $2\pi i \sum_{k,l} w_{h,c}^{k,l} d c_k d \bar{c}_l$.

All this is usually called prequantization data on the pseudokählerian manifold $(M, w_{h,c})$ (See e.q. [11]).

Actually our line bundles $E_{h,c}$ will be analitically trivial, so we identify the total space of $E_{h,c}$ with $M \times \mathbb{C}$ and parametrize it by pairs (f, λ), where f is an univalent function and λ is a complex number.

In [10] the explicit formulae were found for Lie fields defining the action of CVect S^1 on M. Namely, to the vector field $v(e^{it})d/dt$ on S^1 it corresponds the Lie field

$$[\mathcal{L}_V f](z) = -f^2(z) \oint \left[\frac{wf'(w)}{f(w)}\right]^2 \frac{v(w)}{f(w) - f(z)} \frac{dw}{w} \tag{1}$$

REMARK. The variational formulae in the theory of univalent functions obtained by M. Schiffer and G. M. Golusin (see e.g. [6]) look very much like (1): in fact, they correspond to generalized vector fields on $\mathcal{D}_+$ with finite support.

$$\mathcal{L}_V(f, \lambda) = (\mathcal{L}_V f, \lambda\Psi(f, v)) \tag{2}$$

where $\Psi(f, v)$ depends linearly on v and analytically on f.

The condition that (2) gives an action of CVect S^1 or Cvir on $M \times \mathbb{C}$ looks as follows:

(3) $$\delta_u \Psi(f, v) - \delta_v \Psi(f, u) \text{ is a 2 } - \text{ cocycle on } \mathbb{C} \text{ Vect } S^1$$

The simplest expression for $\Psi(f, v)$ satisfying (3) was obtained in [10]:

$$\Psi_0(f, v) = \oint \left[\frac{w f'(w)}{f(w)}\right]^2 v(w) \frac{\mathrm{d}w}{w}$$

It defines an action of CVect S^1 on $M \times \mathbb{C}$.

Yu. A. Neretin discovered the two-parameter family of functions $\Psi_{h,c}(f, v)$ satisfying (3):

(4) $$\Psi_{h,c}(f, v) = h \oint \left[\frac{w f'(w)}{f(w)}\right]^2 v(w) \frac{\mathrm{d}w}{w} + \frac{c}{12} \oint w^2 S(f, w) \frac{\mathrm{d}w}{w}$$

where $S(f, w) = \{f : w\} = \frac{f'''(w)}{f'(w)} - \frac{3}{2}\left[\frac{f''(w)}{f'(w)}\right]^2$ is the Schwartzian (Schwartz derivative) of f with respect to W. For $c \neq 0$ the function $\Psi_{h,c}(f, v)$ defines an action of Cvir on $M \times \mathbb{C}$ such that the central element Z acts on each fiber as a multiplication by ic. For $c = 0\, \Psi_{h,c}$ coincides with $h\Psi_0$.

Let us call a real-valued function $\Phi_{h,c}(f)$ on M an action function corresponding to $\Psi_{h,c}(f, v)$ if the following condition holds:

$$\delta_v \Phi_{h,c}(f) = \operatorname{Re} \Psi_{h,c}(f, v)$$

for all $v \in \text{Vect } S^1$.

The action function corresponding to Ψ_0 was constructed in [10]. At the point $f \in M$ this function is equal to the logarithm of the conformal capacity of the domain $\mathbb{C} \setminus f(\dot{\mathcal{D}}_+)$ with respect to infinity. Unifortunately there are no explicit ways to express this quantity by means of coordinates on M.

The action function $\Phi_{h,c}(f)$ for $c \neq 0$ will be constructed below.

Knowing the action function we can produce an invariant with respect to Cvir hermitian metric in the fibre of the trivial line bundle $M \times \mathbb{C}$ over the point $f \in M$:

$$g_f(\mathrm{d}\lambda, \mathrm{d}\bar{\lambda}) = e^{-\Phi(f)} \mathrm{d}\lambda \mathrm{d}\bar{\lambda}$$

Moreover we can define an invariant hermitian connection on the line bundle $M \times \mathbb{C}$ regarding M as reductive space (see [9], [10]) and using the formulae of Wang (See [12]).

The curvature from $\Omega_{h,c}$ of the connection $\nabla_{h,c}$ can be explicitly expressed in terms of Lie fields $_v$ and we obtain

PROPOSITION 1. $\Omega_{h,c} = 2\pi i w_{h,c}$ ∎

This completes the construction of the so called prequantization bundle $E_{h,c}$ over M. Note also that the action functions $\Phi_{h,c}(f)$ in the Kählerian geometry play the role of potentials to the Kählerian geometry play the role of potentials to the Kähler metrics $w_{h,c}$.

3. MODULES $W_{h,c}$

Our aim here is the description of Cvir – modules arising in the spaces of holomorphic sections of the Lie bundles $E_{h,c}(M)$.

Recall that a point of M is an univalent function $f(z) = z(1 + \sum_k c_k z^k)$. The set of complex numbers $\{c_k\}$ defines an affine coordinate system on M centered in $f_0(z) \equiv z$.

Let us denote by $\mathcal{P}$ the set of all polynomials of $c_1, c_2 \dots$ which have the form

$$P(c_1, \dots, c_N) = \sum_{n=0}^{N} \sum_{\alpha_1,\dots,\alpha_n in \mathbb{Z}_+} A_{\alpha_1,\dots,\alpha_n} c_1^{\alpha_1} \dots c_n^{\alpha_n}$$

$$A_{\alpha_1 \dots \alpha_n} \in \mathbb{C},\ N = 0, 1, 2 \dots$$

Such a polynomial can be viewed as an analytic functional on M. The set $\mathcal{P}$ is the minimal algebra (with respect to the usual multiplication) containing coordinate functions and constants. On the other hand, this set must be dense in the space of all analytic functionals on M in any reasonable topology. So we restrict our considerations here to this class but note that some interesting and important functionals, e.g. the evaluation functionals $f \to f^{(k)}(z)$ for $z \neq 0$ do not belong to $\mathcal{P}$ is just the set of sections $\mathcal{O}(M)$ of the structure sheaf $\mathcal{O}_M$ for the complex quasiaffine manifold M.

We introduce a grading in $\mathcal{O}_M$ by

$$\deg c_1^{\alpha_1} \dots c_n^{\alpha_n} = \sum_{k=1}^{n} k\alpha(k)$$

Note there that the algebra $\mathcal{O}(M)$ depends on the coordinate system chosen and is not conserved by the holomorphic automorphisms of M. In particular, the groups Diff S^1 or Vir do not act on

However, we can easily see that the corresponding infinitesimal action can be defined. Indeed, the formula (1) for the basic elements $e_p \in \mathbb{C}\ \mathrm{Vect}^{pol} S^1$ becomes

$$(5) \qquad (\mathcal{L}_{e_p} f)(z) = -if^2(z) \oint \left[\frac{wf'(w)}{f(w)}\right]^2 \frac{w^{p-1} \mathrm{d} w}{f(w) - f(z)}$$

Denoting $\mathcal{L}_{e_p}$ symply L_p and computing the integral we finally get the following explicit formulae in terms of coordinates $\{c_k\}$ and corresponding derivatives $\partial_k = \partial / \partial c_k$ on M ([10])

$$(6) \qquad \begin{aligned} L_p &= \partial_p + \sum_{k \geq 1} (k+1) c_k \partial_{k+p}, \quad p > 0 \\ L_0 &= \sum_{k \geq 1} (k+1) c_k \partial_k \\ L_{-1} &= \sum_{k \geq 1} [(k+2) c_{k+1} - 2 c_1 c_k] \partial_k \\ L_{-2} &= \sum_{k \geq 1} [(k+3) c_{k+2} + (c_1^2 - 4 c_2) c_k - b_k] \partial_k \end{aligned}$$

where b_k is the Laurent coefficient of $\frac{1}{zf}$ which can be expressed as determinant of the matrix $A^{(k)}$ order $k+2$ with elements

$$a_{ij}^{(k)} = -c_{1+j-i} (c_0 = 1,\ c_k = 0,\ k < 0)$$

These formulae one can view as a definition of the action of $\mathrm{CVect}^{pol} S^1$ in the algebra $\mathcal{O}(M)$. This action does not extend on the whole Lie algebra $\mathrm{CVect}\, S^1$. Note in this connection that in fact in the physical literature only the Lie algebras $\mathrm{CVect}^{pol} S^1$ and Cvir are used even if the terms vector fields or diffeomorphisms are involved.

Further, the Lie algebra $\mathrm{CVect}^{pol} S^1$ is naturally graded: $\deg e_p = p$. It is easy to see from (6) that the action agrees with this grading.

We proceed now to the analytic bundles $E_{h,c}(M)$. As the formal analog of the space $\Gamma_{\mathcal{O}}(M, E_{h,c})$ of holomorphic sections of a free $\mathcal{O}_{(M)}$-module $W_{h,c}$ arises. The trivialisation of $E_{h,c}$ in the algebraic language means the identification $W_{h,c}$ with $\mathcal{O}(M)$ by the choice of a free generator. We shall use trivialisation obtained in the previous section.

Recall that the Lie field corresponding to $(v, \tau) \in \mathbb{C}\mathrm{vir}$ at the point $(f, \lambda) \in E_{h,c}(M)$ is given by the equalities

$$[\ {}_{v,\tau}f](z) = -if^2(z) \oint \left[\frac{wf'(w)}{f(w)}\right]^2 \frac{v(w)}{f(w)-f(z)} \frac{\mathrm{d}w}{w}$$

(7)
$$ {}_{v,\tau}\lambda = \{h \oint \left[\frac{wf'(w)}{f(w)}\right]^2 v(w) + \frac{\mathrm{d}w}{w} + $$
$$ + \frac{c}{12} \oint wS(f,w)\mathrm{d}w + ic\tau\}\lambda $$

Repeting the same procedure as for $\mathcal{O}(M)$ we get the explicit formulae for the action of Cvir on $W_{h,c}$:

(8)
$$\begin{aligned}
\tilde{L}_p &= \partial_p + \sum_{k\geq 1}(k+1)c_k\partial_{k+p}, \quad p>0 \\
\tilde{L}_0 &= \sum_{k\geq 1} kc_k\partial_k + h \\
\tilde{L}_{-1} &= \sum_{k\geq 1}[(k+2)c_{k+1} - 2c_1c_k]\partial_k + 2hc_1 \\
\tilde{L}_{-2} &= \sum_{k\geq 1}[(k+3)c_{k+2} + (c_1^2 - 4c_2)c_k - b_k]\partial_k + \\
&\quad + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2) \\
\tilde{Z} &= c.
\end{aligned}$$

PROPOSITION 2. $W_{h,c}$ *is graded Cvir-module via the action given by (8).* ∎

The important role in the study of infinite dimensional Lie algebras and their modules is played by the so called singular vectors, i.e. invariant vectors with respect to some subalgebras.

We are interested here in the singular vectors with respect to subalgebra $\mathcal{L}_+ =$ Lin $\{\tilde{L}_p, p\geq 1\}$. The polynom $P(c_1\dots c_n)$ is a singular vector for $\mathcal{L}_+$ if for each integer $p>0$ the equality holds

$$\left[\partial_p + \sum_{k\geq 1}(k+1)c_k\partial_{k+p}\right] P(c_1\dots c_n) = 0$$

PROPOSITION 3. *The only singular vector in $W_{h,c}$ for $\mathcal{L}_+$ is a constant. We have*

$$\tilde{L}_0 \cdot 1 = h \cdot 1, \tilde{Z} \cdot 1 = c \cdot 1.$$

The submodule $L_{h,c} \subset W_{h,c}$ generated by 1 is irreducible and has the highest weight (h, c).

Proof. Consider the universal enveloping algebra U of the Lie algebra Cvir. It is clear that $L_{h,c} = U \cdot 1$. Denote by $W^*_{h,c}$ the dual graded $\mathcal{O}(M)$-module to $W^*_{h,c}$. By definition $W^*_{h,c}$ is the linear span of $\left(W^{(n)}_{h,c}\right)^*$ where $W^{(n)}_{h,c}$ is the homogeneous component of $W_{h,c}$ of degree n.

The linear space $W^*_{h,c}$ can be identified with $\mathcal{O}(M)$ by the pairing

$$\langle P, Q\rangle = P(\partial_1 \dots \partial_n)Q(c_1 \dots c_n)|_{c_1=\dots c_n=0}$$

The Lie algebra Cvir acts on $W^*_{h,c}$

$$\tilde{L}^*_p = -c_p - \sum_{k\geq 1}(k+1)c_{k+p}\partial_k, \quad p > 0$$

$$\tilde{L}^*_0 = -\sum_{k\geq 1} kc_k\partial_k - h$$

(9)
$$\tilde{L}^*_{-1} = -\sum_{k\geq 1}(k+2)c_k\partial_{k+1} + 2\sum_{k\geq 1} c_k\partial_1\partial_k - 2h\partial_1$$

$$\tilde{L}^*_{-2} = -\sum_{k\geq 1}[(k+3)c_k\partial_{k+2} + (c_1^2 - 4c_2)c_k - b_k(\partial_1, \dots, \partial_n)] + h(\partial_1^2 - 4\partial_2) + \frac{c}{2}(\partial_1^2 - \partial_2),$$

$$\tilde{Z}^* = -c.$$

The singular vectors in $W^*_{h,c}$ for the subalgebra $\mathcal{L}_- = \mathrm{Lin}(L_p, p < 0)$ are determined by the equations

$$\tilde{L}^*_{-1}v = \tilde{L}^*_{-2}v = 0.$$

In particular the constant 1 is a singular vector of the weight $(-h, -c)$. ■

PROPOSITION 4. *$W^*_{h,c}$ is a free $U(\mathcal{L}_-)$-module. The substitution $L_p \to -L_p, Z \to -Z$ turn it into the Verma module $V_{h,c}$ ([13]). The module $W_{h,c}$ is irreducible iff $V_{h,c}$ is irreducible ([14]). Let $1, v_1, \dots$ be the total set of singular vectors in $V_{h,c}$. Then the irreducible submodule $L_{h,c}$ coincides with $\mathrm{Ann}_{W_{h,c}} U(1, v_1 \dots)$.* ■

4. THE IMBEDDING OF THE MANIFOLD M IN THE INFINITE DIMENSIONAL SYMMETRIC DOMAIN OF TYPE III

The study of finite dimensional Kähler manifold is reduced to a great extent to the study of the rather narrow class of objects - complex projective spaces. The universal character of these spaces is caused by the three statements: the Kodaira theorem about imbedding of any compact manifold into a projective space, the Calabi theorem about isometric imbedding of Kähler manifolds into (possibility infinite dimensional or with non-definite metric) projective spaces, and the Borel-Weil-Bott theorem about G-equivariant imbedding of a G-homogeneous compact Kähler manifold onto projective space over the hightest weight module for a compact Lie group G.

Unfortunately the first two theorems do not give the explicit construction of imbedding and the last theorem supposes the detailed information about the structure of highest weight modules.

Nevertheless for a certain class of Kähler manifolds there are known the explicit ways of imbedding in projective spaces. In particular this class contains grassamanians for classical Lie groups and their non-compact forms – the classical symmetric domains.

If an imbedding of a Kähler manifold in such a manifold is known we can imbed it in the projective space. This consideration remains true in the infinite dimensional situation.

Our manifold M can be imbedded in many ways in the infinite dimensional analogs of classical domains (see [15]). We consider the imbedding in the infinite dimensional analog of the symmetric domain of type III in the terminology of E. Cartan.

Let H denote the space of smooth real 1-forms $u(e^{i\theta})d\theta$ on the unit circle with zero integral over the circle. Let $H^{\mathbb{C}}$ denote the complexification of H and $H^{\mathbb{C}}_{\pm}$ be the transversal subplace of $H^{\mathbb{C}}$ consisting of forms $u(e^{i\theta})d\theta$ admitting an analytic continuation in $\mathcal{D}_{\pm}$ (i.e. inside or outside the circle). We can identify $H^{\mathbb{C}}$ with $\mathcal{O}(S^1)/\,\text{Const.}$ via $f \to df$. Then $H^{\mathbb{C}}_{\pm}$ corresponds to $\mathcal{O}(\mathcal{D}_{\pm})/\,\text{Const.}$

The general element of the group $GL(H^{\mathbb{C}})$ has the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where } \begin{array}{ll} A : H^{\mathbb{C}}_{+} \to H^{\mathbb{C}}_{+}, & B : H^{\mathbb{C}}_{-} \to H^{\mathbb{C}}_{+} \\ C : H^{\mathbb{C}}_{+} \to H^{\mathbb{C}}_{-}, & D : H^{\mathbb{C}}_{-} \to H^{\mathbb{C}}_{-}. \end{array} \tag{10}$$

The subgroup $GL(H)$ is distinguished by the conditions $D = \bar{A}, C = \bar{B}$.

On the space $H^{\mathbb{C}}$ we define the sympletic and the pseudohermitian structures posing

$$\begin{aligned} \langle f|g\rangle &= \oint f\,dg \\ (f|g) &= \oint f\,\overline{dg}, \quad f, g \in \mathcal{O}(S^1) \end{aligned} \tag{11}$$

Denote by $Sp(H^{\mathbb{C}})$ and $U(H^{\mathbb{C}}_{+}, H^{\mathbb{C}}_{-})$ the groups preserving these structures and by $Sp(H, \mathbb{R})$ their common part which is also the intersection of $Sp(H^{\mathbb{C}})$ with $GL(H)$.

The elements of $Sp(H^{\mathbb{C}})$ are characterised be the conditions

$$A'C = C'A, B'D = D'B, A'D - C'B = 1$$

and the elements of $Sp(H\mathbf{R})$ by the conditions

$$A^*A - C^*C = 1, C'A = A'C, B = \bar{C}, D = \bar{A}.$$

Let $Gr(H^{\mathbb{C}})$ be the set of all complex Lagrangean subspaces in $H^{\mathbb{C}}$. It is an infinite dimensional manifold with $Sp(H^{\mathbb{C}})$ as a group of motions. The stabiliser of the point $H_-^{\mathbb{C}} \in Gr(H^{\mathbb{C}})$ is the subgroup F of matrices of the form (10) with $B = 0$.

Let us consider the action of $Sp(H\mathbf{R})$ on $Gr(H^{\mathbb{C}})$. The orbit passing through the point $H_-^{\mathbb{C}}$ is manifold $\mathcal{R}$ isomorphic as homogeneous space to $Sp(H^{\mathbb{C}})/U$ where U consists of matrices $\begin{pmatrix} A & O \\ O & \bar{A} \end{pmatrix}$, $A \in U(H_+^{\mathbb{C}})$.

The manifold $\mathcal{R}$ is open in $Gr(H^{\mathbb{C}})$ and is an infinite dimensional analog of a classical domain of type III. ([17]), which plays a significant role in the geometric quantization ([11]). It can be imbedded in Hom $(H_-^{\mathbb{C}}, H_+^{\mathbb{C}})$ as the set of symmetric matrices Z satisfying the condition $1 - Z\bar{Z} > 0$. The action of $Sp(H, \mathbf{R})$ on $Z \in \mathcal{R}$ is written by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}$$

The skeleton (Shilov boundary) of $\mathcal{R}$ consists of unitary symmetric matrices Z.

The manifold $\mathcal{R}$ is an infinite dimensional Kähler manifold in the following sence. Let $\mathcal{R}_0 = \mathcal{R} \cap \mathcal{HS}(H_-^{\mathbb{C}}, H_+^{\mathbb{C}})$ where $\mathcal{HS}$ means Hilbert-Schmidt operators. Then $\mathcal{R}_0$ is a dense set in $\mathcal{R}$ and homogeneous manifold with respect to the so called restricted sympletic group $Sp(H, \mathbf{R})$ consisting of matrices of the ferm (10) with B and C being Hilbert-Schmidt and A, D-Fredholm operators ([18]). On the tangent space to R_0 at the point $Z = 0$ we define a Kähler metric $h(\delta_1 Z, \delta_2 Z) = \operatorname{tr}(\delta_1 Z, \delta_2 \bar{Z})$ Its potential is equal to log det $(1 - Z\bar{Z})$.

To each $Z \in \mathcal{R}_0$ we can associate a function $f(z, w)$ of Hardy class $H^2(\mathcal{D}_+ \times \mathcal{D}_+)$:

$$U(z, w) = \sum_{k,l \geq 1} Z_{kp} z^k w^l \tag{12}$$

and a Fredholm operator $T = 1 - K$ where K is an integral operator on the Hardy space $H^2(\mathcal{D}_+)$ with the kernel function $K(z, \bar{w})$ defined by

$$K(z, \bar{w}) = \oint_{|t|=1} U(z, t)\, \overline{\mathrm{d}_t U(t, w)}$$

Denote by $\Phi(U)$ the quantity Log det T. By the Fredholm perturbation formula ([19]) we have

$$\Phi(U) = \log\left[1 + \sum_{k\geq 1}(-1)^k \oint_{|z_i|=1} \oint \det \|\mathrm{d}_{z_i} K(z_i, \bar{z}_j)\|_{i,j\leq k}\right] \tag{13}$$

We describe now the imbedding of the manifold M in the domain $\mathcal{R}_0$. The natural representation of $\mathrm{Diff}_+ S^1$ in the space $H^{\mathbb{C}}$ defines a monomorphis $\mathrm{Diff}_+ S^1 \to Sp_0(H, \mathbf{R})$ (see [15]). Hence, the group $\mathrm{Diff}_+ S^1$ acts on $\mathcal{R}_0$. The stabilizer of the point $0 \in \mathcal{R}_0$ in $\mathrm{Diff}_+ S^1$ coincides with the subgroup $H_1 = PSL(2, \mathbf{R})$ ([10]). Thus we obtain a mapping of M into $\mathcal{R}_0$ which factors through the natural projection $\Pi_1 : M \to M_1$ where $M_1 = \mathrm{Diff}_+ S^1/H_1$ is a factor manifold of M with respect to the first Ovsienko's foliaction ([9], [10]).

More precisely, to the univalent function $f(z)$ representing a point of M we associate a subspace $V_f = \{zf'(z)f^{-1-k}(z)\mathrm{d}\theta | k \geq 0\} \cup H^{\mathbb{C}} \in Gr(H^{\mathbb{C}})$. (We use here the relation $\gamma = f^{-1} \circ g$ from [8] and the fact that $H^{\mathbb{C}}$ is generated by $g^{-k}\mathrm{d}g$, $k \geq 2$).

The following two lemmas can be proved by direct computations:

LEMMA 1. *The matrix Z_f corresponding to the subspace $V_f \in Gr(H^{\mathbb{C}})$ belongs to the Hilbert-Schmidt class and has elements*

$$(Z_f)_{mn} = \sqrt{\frac{\bar{m}}{n}} \det \begin{bmatrix} 1 \\ \vdots \\ z^{n-1} \\ z^{-\bar{m}} \end{bmatrix} \left[\frac{1}{f} \cdots \frac{1}{f^{n+1}}\right] \mathrm{d}z$$ ■

COROLLARY. *The matrix coefficients of Z_f belong to the ring $\mathcal{O}(M)$.* ■

LEMMA 2. *The operator Z_f maps $Z^{-m} \in \mathcal{O}(\mathcal{D}_-)$ into $\Phi_m(1/f) - Z^m \in \mathcal{O}(\mathcal{D}_+)$ where Φ_m denotes the m-th Faber polynom for the function $f^{-1}(z^{-1})$.* ■

The details about Faber polynoms one can see in [21]. The matrix Z_f had appeared first in the paper by Grunsky [22] in 1939.

Recently the mapping $f \to V_f$ was rediscovered in the soliton theory and was extended to arbitrary Riemann surfaces by Krichever [23]. It is known now as « Krichever's construction» [18]. The condition $1 - Z_f \bar{Z}_f > 0$ characterizes the class S of univalent functions. It reduces to the countable family of scalar inequalities which are called Grunsky conditions for univalence.

The function U_f of the Hardy class $H^2(\mathcal{D}_+ \times \mathcal{D}_+)$ associated to the matrix Z_f via [10] has the form

$$U_f(z,w) = \log \frac{1/f(z) - 1/f(w)}{1/z - 1/w}$$

(cf. also [6], [24]).

It is interesting to remark that the orthogonality conditions for Faber polynoms obtained by I. M. Milin [25] using the area method imply that Z_f belongs to the skeleton of the boundary of $\mathcal{R}_0$ iff $f \in S^*$, i.e. mes $(\mathbb{C}\backslash(\dot{(\mathcal{D}_+)})) = 0$. The set S^* is not $\mathrm{Diff}_+ S^1$-transitive. It contains as a proper subset the sceleton of the boundary of M which is isomorphic to $\mathrm{Diff}_+ S^1/K$ where K is the centralizer of the involution $\delta : z \to \bar{z}$ of the circle. (See [26], [27]).

Finally we have

PROPOSITION 5. The Kähler metric on M induced from the above mapping $M \to \mathcal{R}_0$ coincides with $w_{0,1}$. Its potential has the form (13) where

$$K(z,\bar{w}) = \oint_{|t|=1} \log \frac{1/f(z) - 1/f(t)}{1/z - 1/t}\, d_t \log \overline{\frac{1/f(t) - 1/f(w)}{1/t - 1/w}} \tag{14}$$

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